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Uniformly valid solution of the Orr–Sommerfeld equation by a modified Heisenberg method

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The classical Heisenberg method of solving the Orr–Sommerfeld equation is modified in such a way that inner and outer expansions are replaced by a uniformly valid successive approximation in which no data on the second derivative of the parallel shear profile are needed. It is shown that this feature enables us to calculate stability characteristics for wider classes of flows with improved accuracy. As a preliminary check for validity of the method, stability of the Blasius flow is calculated and compared with existing methods. It turns out that the method works for high Reynolds numbers, up to about 10^5, and that the expressions for the eigenfunctions and the eigenvalue condition are much simpler than those found by existing methods.

1. Introduction

As is well known, analytical methods of solving the Orr–Sommerfeld equation have their origin in the pioneering work of Heisenberg (1924) and underwent later development by Lin (1945, 1946) suitable for applications to practical problems. The approach along this line has provided us with a number of fruitful assertions on the nature of the flow stability (see e.g. Drazin and Reid 1981).

On the other hand, powerful computational methods have been developed with the advent of high-speed computers and seem to have outgrown other methods of solution, at least in achieving accuracy (Betchov & Criminale 1967). We can calculate, then, eigenvalues and eigenfunctions of the equation with sufficient accuracy if data on the velocity profile are prescribed up to the second derivative, and if the Reynolds number is not too high.

These two approaches are, in principle, complementary to each other in the sense that the analytical method yields better results for high-Reynolds-number cases in view of its nature as the asymptotics, whereas the computational methods favour low Reynolds numbers because of the inherent problem of step-size limitations. The actual situation, however, is that, even for the simplest case of Blasius flow, the high-Reynolds-number limit of the computational methods is still too low to allow for smooth matching with asymptotic curves calculated by Lin (Drazin & Reid 1981).

In order to bridge this gap, we should have a theory in which the concept of asymptotics is abandoned, and which has a wider region of validity covering lower Reynolds numbers. This, in turn, requires the retention of the viscous term in the governing equation throughout the region under consideration.

This problem has been motivated by an attempt to improve the classical Heisenberg formalism so as to eliminate the so-called 'patching' procedure caused by the failure of asymptotic matching. In solving the Orr–Sommerfeld equation

$$\left[(u-c)(\mathrm{D}^2-A^2)-\frac{1}{\mathrm{i}\alpha R}(\mathrm{D}^2-A^2)^2-u''\right]Y=0, \tag{1.1}$$

with $$\mathrm{D} \equiv \frac{\mathrm{d}}{\mathrm{d}y}, \quad u' \equiv \mathrm{D}u$$

(where R, c, A and α denote respectively the Reynolds number, the wave-propagation velocity, the total wavenumber and its streamwise component), Heisenberg ignored the terms with the small parameter

$$\epsilon' = (\alpha R)^{-1} \tag{1.2}$$

except in the vicinity of the critical point,

$$|u-c| < \epsilon'. \tag{1.3}$$

Then the resulting inviscid equation

$$[(u-c)(\mathrm{D}^2-A^2)-u''] Y = 0 \tag{1.4}$$

is shown to be integrable analytically through expanding Y in terms of A^2 as

$$Y = \sum_{n=0}^{\infty} A^{2n} Y_n. \tag{1.5}$$

This solution is supposed to be connected with the so-called viscous solution valid in region (1.3). Unfortunately (though not fatally), continuation of the two solutions does not conform with the rule of matching required by asymptotic analysis.

Reexamining the Heisenberg method, Tsugé (1978) has pointed out that the analysis can be carried through without deleting the viscous term if the expansion (1.5) is employed. In fact, then, the equation governing

$$Y_0 = \Phi \tag{1.6}$$

is, from (1.1) and (1.5),

$$\left[\frac{\mathrm{D}^4}{\mathrm{i}\alpha R} - (u-c)\,\mathrm{D}^2 + u''\right]\Phi = 0, \tag{1.7}$$

which is integrated once immediately, yielding the third-order equation

$$\left[\frac{\mathrm{D}^3}{\mathrm{i}\alpha R} - (u-c)\,\mathrm{D} + u'\right]\Phi = C_1. \tag{1.8}$$

This equation is seen to have the following favourable features. First, a third-order equation is much easier to handle than a fourth-order equation. Actually, if we can manage to get a particular solution, we are then left with a second-order equation for which versatile mathematical tools are available to obtain the remaining solutions. Secondly, by virtue of the presence of the viscous (third-order) term, the point $u-c = 0$ ceases to be a critical point, in contrast with (1.4). The singularity consideration that had complicated the classical analysis is no longer necessary. Thirdly, the curvature term disappears through the integration, and is replaced by the first derivative. The advantage of eliminating the u'' term in the equation is obvious, in particular for flows having no analytical expression for $u(y)$. Employing the adjoint Orr–Sommerfeld equation instead of the original one has the same merit.

A few remarks must be made regarding the reason for not employing the usual expansion in a small parameter in the proposed method. This situation is examined by introducing a scaling

$$y = \epsilon\eta, \tag{1.9}$$

with $$\epsilon = (\alpha R)^{-\frac{1}{2}}.$$

In terms of the stretched variable η, (1.8) is transformed as

$$\frac{\dddot{\Phi}}{\mathrm{i}} - (u-c)\,\dot{\Phi} + \dot{u}\Phi = C', \tag{1.10}$$

where $\dot{\Phi} \equiv \mathrm{d}\Phi/\mathrm{d}\eta$. Note that the stretching rate ϵ^{-1} is greater here than that used in the classical theory, namely $(\epsilon'')^{-1} \sim (\alpha R)^{\frac{1}{3}}$. We observe that the last term on the left-hand side of the equation is smaller by a factor of ϵ compared with the other terms, because u varies with y, whereas the differentiation is with respect to η. If we deleted the term for this reason, the classical 'viscous' solution would result, and $\Phi(\eta)$ would be expressed in terms of Hankel functions. On the other hand, correct far-field behaviour of the solution is secured only through balance of the last two terms on the left-hand side (Heisenberg 1924). This means that $\dot{\Phi}$ varies like $\dot{u}$, in other words, that Φ is a function of y rather than η at large distances from the viscous layer, where the first two terms compete. It is then obvious that, as far as the calculation is practicable, it is preferable to work with the full equation (1.8) without invoking any scaling or approximations associated with it. In fact, retention of these three terms is the key issue of the uniform validity of solution by which we are able to eliminate the procedure such as inner or outer expansions of the asymptotics.

The approach to the solution is as follows. Provided that we have obtained all the solutions in the first approximation of (1.8), a routine procedure of successive approximation will improve the accuracy of the solution automatically. Therefore the central part of the analysis will be to derive four independent solutions of (1.8), which we shall discuss in §2. Emphasis will be on differences as well as similarities of each solution compared with its classical counterpart due to Heisenberg and Lin. Section 3 will be devoted to discussions of the higher approximation, where a technique is introduced to speed up the convergence. The eigenvalue problem is formulated (§4) by using those elementary solutions, and the proposed method is compared with the existing one through calculated stability characteristics of the Blasius boundary-layer profile.

2. Analytical background of the solution of (1.8)

In this section we are concerned with obtaining the four independent solutions of (1.8), namely

$$\epsilon\Phi''' - \mathrm{i}(u-c)\,\Phi' + \mathrm{i}u'\Phi = \mathrm{i}C_1. \tag{2.1}$$

In numbering the four solutions, we have followed conventional usage (see Lin 1945, 1946) so that each counterpart has the same asymptotic behaviour; Φ_1 and Φ_2 tend to moderately varying functions, whereas Φ_3 and Φ_4 exhibit exponential decay and growth respectively. Our procedure of obtaining the solutions will be carried out in the order of $\Phi_3 \rightarrow \Phi_2 \rightarrow \Phi_4 \rightarrow \Phi_1$ in such a form that each solution will be obtained using knowledge of Φs at previous stages, and Φ_4 and Φ_1 will be solved as exact solutions in terms of Φ_3 and Φ_2.

Solution Φ_3

First we consider the homogeneous equation ($C_1 = 0$) of (2.1), which provides three independent solutions Φ_2 through Φ_4:

$$\epsilon\Phi''' - \mathrm{i}(u-c)\,\Phi' + \mathrm{i}u'\Phi = 0. \tag{2.2}$$

This equation has a structure such that the critical layer ($u = c$) is an ordinary point, so that no artifice as needed in the classical treatment is necessary in integrating the equation across the point. In view of this feature, and of the prospective exponential decay, the solution Φ_3 may be assumed in the form

$$\Phi_3 = \exp \int_0^y \lambda \, \mathrm{d}y. \tag{2.3}$$

Then (2.2), subject to this transformation of the dependent variable, reads

$$\epsilon(\lambda'' + 3\lambda\lambda' + \lambda^3) - \mathrm{i}(u-c)\,\lambda + \mathrm{i}u' = 0.$$

This equation reduces to first-order simultaneous equations by introducing an additional intermediate variable μ,

$$\lambda' = \mu - \mathrm{i}\epsilon^{-1}(u-c), \quad \mu' = -3\lambda\mu + 4\mathrm{i}\epsilon^{-1}(u-c)\,\lambda - \lambda^3. \tag{2.4}$$

Since we are seeking a solution of (2.4) that *decays* exponentially in the asymptotic limit $(\lambda', \mu' \sim 0)$, the above equations require the root λ with negative real part:

$$\left.\begin{aligned} \lambda &\sim -\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\,\epsilon^{-\frac{1}{2}}(u-c)^{\frac{1}{2}}, \\ \mu &\sim \mathrm{i}\epsilon^{-1}(u-c) \end{aligned}\right\} \quad \text{for } y \gg 1. \tag{2.5}$$

The system of equations (2.4) can be transformed into a simple form in the following way. Multiply the first equation $\alpha\lambda$ and then add it to the second one. Then we have

$$\frac{\alpha}{2}\left[\lambda^2 + \frac{2}{\alpha}\mu + \frac{2\beta}{\alpha}\frac{\mathrm{i}(u-c)}{\epsilon}\right]' + \lambda\left[\lambda^2 + (3-\alpha)\,\mu + (\alpha-4)\frac{\mathrm{i}(u-c)}{\epsilon}\right] = \mathrm{i}\beta\frac{u'}{\epsilon},$$

where terms with β have been introduced on both sides without violating the equality. Parameters α and β are determined by the condition that quantities in the square brackets take the same value. Two sets of values (α, β) to meet this condition are found to be $(1, -\frac{3}{2})$ and $(2, -2)$. This is equivalent to introducing the variable-transformation

$$V = -\mu + 2\mathrm{i}\epsilon^{-1}(u-c) - \lambda^2, \quad U = -2\mu + 3\mathrm{i}\epsilon^{-1}(u-c) - \lambda^2 \tag{2.6}$$

and to work with the equations in the new dependent variables (V, U):

$$V' + \lambda V - 2\mathrm{i}\epsilon^{-1}u' = 0, \quad U' + 2\lambda U - 3\mathrm{i}\epsilon^{-1}u' = 0, \tag{2.7}$$

subject to the conditions

$$V', U' \sim 0 \quad \text{as } y \to \infty. \tag{2.8}$$

The actual integration may be started at a point $y = \hat{y}(\gg 1)$ with initial values

$$V = \frac{2\mathrm{i}\epsilon^{-1}u'}{\lambda}, \quad U = \frac{3\mathrm{i}\epsilon^{-1}u'}{2\lambda}, \tag{2.9}$$

where λ, given by (2.5), is moderately small. The quadrature marches inward from this point, with the nonlinearity of (2.7), appearing in the form

$$\lambda = -\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\,[\epsilon^{-1}(u-c) + 2\mathrm{i}\,V - \mathrm{i}\,U]^{\frac{1}{2}} \tag{2.10}$$

(as obtained from (2.6) by eliminating μ), taken into account for $y < \hat{y}$. Outside this region we have

$$\lambda = -\frac{1+\mathrm{i}}{(2\epsilon)^{\frac{1}{2}}}(u-c)^{\frac{1}{2}} - \frac{5}{4}\frac{u'}{u-c}, \tag{2.11}$$

where the asymptotic form (2.9) has been employed. Actually the point $y = \hat{y}$ is so chosen at to make the second term of (2.11) small by a factor of 10^{-3} compared with the first term.

It should be remarked here that this method of numerical integration of the original equation (2.4) works successfully only for the decaying solution Φ_3, and that a formally identical procedure for the growing solution Φ_4 suffers from numerical instabilities. The difference has its origin in the fact that the point $y = \infty$ is a saddle singularity of (2.4) for the case treated, whereas in the Φ_4 case it proves to be a nodal singularity from which an infinite number of solutions emerge.

It can be demonstrated that our solution (2.3) has the same asymptotic form as its classical equivalent for $y > \hat{y} \gg 1$. In fact, a straightforward calculation of (2.3) with (2.11) substituted therein leads to

$$\Phi_3(y) = \Phi_3(\hat{y})\left(\frac{\hat{u}-c}{u-c}\right)^{\frac{5}{4}} \exp\left[-e^{\frac{1}{4}\pi i}\,\epsilon^{-\frac{1}{2}}\int_{\hat{y}}^{y}(u-c)^{\frac{1}{2}}\,dy\right] \quad \text{for } y > \hat{y} \gg 1. \tag{2.12}$$

On the other hand, the classical counterpart of Φ_3 is

$$[\Phi_3(\tau)]_{CL} = \int_{\infty}^{\tau} d\tau \int_{\infty}^{\tau} d\tau\, \tau^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}[\tfrac{2}{3}(ia_0\tau)^{\frac{3}{2}}], \tag{2.13}$$

as given by Lin, where $H_{\frac{1}{3}}^{(1)}$ denotes the Hankel function of the first kind, and τ and a_0 are defined by

$$\tau = (y-y_c)(\alpha R)^{\frac{1}{3}}, \quad a_0 = (u_c')^{\frac{1}{3}}, \tag{2.14}$$

with subscript c signifying the value at the critical point. This solution, corrected to hold at far field, is given by Tollmien's improved viscous approximation (Drazin & Reid 1981, p. 176), and its asymptotic expression coincides with (2.12) to within a multiplicative constant.

Solution Φ_2

Once we have obtained one of the solutions of the third-order equation (2.2), difficulties are considerably lessened in solving for the rest of them because of the following theorem. If n independent solutions of a mth-order linear differential equation are available, the $(n+1)$th solution is obtained by solving an equation of $(m-n)$th order. This theorem applied to the current case for the second solution Φ_2 needs an equation of order $3-1 = 2$ to be obtained by putting

$$\Phi_2 = \Phi_3 \int_0^y G\,dy \tag{2.15}$$

and by substituting this into (2.2). This, in turn, is equivalent to claiming that, if G is assumed in the form

$$G = \exp \int_0^y (-\lambda + T)\,dy, \tag{2.16}$$

the transformed variable T obeys a first-order (nonlinear) equation. A simple calculation actually confirms the assertion, leaving us with a Riccati equation

$$T' + T^2 + \lambda T - U = 0, \tag{2.17}$$

where λ and U have been obtained from (2.10) and (2.7) respectively. Of the two asymptotic roots of (2.17) for $T \sim 0$, the one that vanishes as $y \to \infty$, namely

$$T = U/\lambda, \tag{2.18}$$

turns out to be the correct choice. This root alone given rise to the solution Φ_2 meeting the requirement of moderate variation as $y \to \infty$. In fact, then

$$\begin{aligned} \Phi_2 &= \Phi_3 \int_0^y dy \exp\left(\int_0^y (-\lambda + T)\,dy\right) \\ &= \hat{B}_2(u-c) \quad \text{for } y > \hat{y}, \end{aligned} \tag{2.19}$$

with $$\hat{B}_2 = \epsilon^{-\frac{1}{2}}(\hat{u}-c)^{-\frac{3}{2}} \exp\left(-\tfrac{1}{4}\pi\mathrm{i} + \int_0^{\hat{y}} T\,\mathrm{d}y\right),$$

where use is made of the fact that the integrand in (2.19) is a rapidly growing function to compensate for the rapid decay of Φ_3, so that only the region in the vicinity of the upper bound contributes to the integral; the integral may then be replaced with

$$(-\lambda+T)^{-1}\exp\int_0^y (-\lambda+T)\,\mathrm{d}y.$$

It seems to be of interest to compare Φ_2 with its classical counterparts

$$(\Phi_2)_{\mathrm{CL,\,in}} = 1+O(\alpha R)^{-\frac{1}{3}}, \quad (\Phi_2)_{\mathrm{CL,\,out}} = u-c \tag{2.20}$$

representing inner (viscous) and outer (inviscid) Heisenberg solutions respectively. We note that the classical Φ_2 is plainly slowly varying, whereas Φ_2 as given here is moderately varying in the interior region and tends only asymptotically ($y \gg 1$) to a slowly varying function (2.19). Also to be noted is the coincidence of (2.19) with the leading term of the classical outer solution, proving a sound basis of the asymptotic scheme adopted here.

For later use we note that Φ_2 obeys the following first-order differential equation:

$$\frac{\mathrm{d}\Phi_2}{\mathrm{d}y} = \lambda\Phi_2 + S, \tag{2.21}$$

with $$S = \exp\int_0^y T\,\mathrm{d}y, \tag{2.22}$$

the validity of which is easily checked by differentiating the expression (2.15) and substituting (2.16) and (2.3) into the result.

Solution Φ_4

Repeated use of the foregoing theorem assures that the third solution Φ_4 satisfies a first-order linear differential equation to be deduced from (2.2). This assertion can be materialized by means of the method of variation of constants:

$$\Phi_4 = \Gamma_2\Phi_2 + \Gamma_3\Phi_3, \tag{2.23}$$

where Φ_2 and Φ_3 are functions of y to be determined. Having eliminated Γ_2 between the original equation (2.2) and the supplementary condition

$$\Gamma_2'\Phi_2 + \Gamma_3'\Phi_3 = 0$$

standard in this method, we are led to an equation for Γ_3 that is essentially of the first order:

$$\frac{\Gamma_3''}{\Gamma_3'} = \frac{\Phi_2'}{\Phi_2} - 2\frac{\Phi_2\Phi_3'' - \Phi_3\Phi_2''}{\Phi_2\Phi_3' - \Phi_2'\Phi_3}.$$

This equation is analytically integrated to give

$$\Phi_4 = -\Phi_2\int_0^y \Phi_3 w^{-2}\,\mathrm{d}y + \Phi_3\int_0^y \Phi_2 w^{-2}\,\mathrm{d}y, \tag{2.24}$$

where w is the Wronskian formed by Φ_2 and Φ_3:

$$w = \begin{vmatrix} \Phi_2 & \Phi_3 \\ \Phi_2' & \Phi_3' \end{vmatrix}. \tag{2.25}$$

It is easily confirmed from the analytical solution (2.24) for Φ_4 that this solution has an asymptotic form of exponential growth with y. In fact, using the same approximation as has been utilized in deriving (2.19), we have for (2.24)

$$\left.\begin{aligned} \Phi_4 &\sim \frac{1}{2}\frac{1}{\lambda^3 \Phi_2 \Phi_3} \sim \hat{B}_4 (u-c)^{-\frac{5}{4}} \exp\left[\mathrm{e}^{\frac{1}{4}\pi \mathrm{i}} \int_{\hat{y}}^{y} (u-c)^{\frac{1}{2}}\,\mathrm{d}y\right], \\ \text{with}\qquad \hat{B}_4 &= \frac{1}{2}\frac{\mathrm{e}^{\frac{1}{4}\pi \mathrm{i}}}{\hat{B}_2 \hat{\Phi}_3}\frac{1}{(\hat{u}-c)^{\frac{5}{4}}}, \end{aligned}\right\} \tag{2.26}$$

where, in deriving the second line, the asymptotic expressions (2.12) and (2.19) for Φ_2 and Φ_3 have been used. Relationship (2.26) is again in agreement with its classical equivalent, as in the case of the decaying solution Φ_3. The classical equivalent is

$$[\Phi_4(\tau)]_{\mathrm{CL}} = \int_{-\infty}^{\tau} \mathrm{d}\tau \int_{-\infty}^{\tau} \mathrm{d}\tau\, \tau^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}[\tfrac{2}{3}(\mathrm{i}a_0 \tau)^{\frac{3}{2}}], \tag{2.27}$$

with the same nomenclatures as before and with the Hankel function of the second kind $H_{\frac{1}{3}}^{(2)}$. The same far-field correction as given to Φ_3 is applicable also here, and the result shows agreement with (2.26).

The set of solutions (Φ_2, Φ_3, Φ_4) thus constructed has noteworthy characteristics for the Wronskian, which serve to simplify the analyses to follow:

$$w_{\mathrm{III}} \equiv \begin{vmatrix} \Phi_2 & \Phi_3 & \Phi_4 \\ \Phi_2' & \Phi_3' & \Phi_4' \\ \Phi_2'' & \Phi_3'' & \Phi_4'' \end{vmatrix} = 1. \tag{2.28}$$

This formula is easily checked by noting the following relationship:

$$\frac{\mathrm{d}^l \Phi_4}{\mathrm{d}y^l} = -\frac{\mathrm{d}^l \Phi_2}{\mathrm{d}y^l}\int_0^y \frac{\Phi_3}{w^2}\mathrm{d}y + \frac{\mathrm{d}^l \Phi_3}{\mathrm{d}y^l}\int_0^y \frac{\Phi_2}{w^2}\mathrm{d}y + \frac{\delta_{l2}}{w}, \quad l = 0, 1, 2, \tag{2.29}$$

where δ is the Kronecker delta.

Solution Φ_1

Since all the homogeneous solutions now have been exhausted, the fourth solution Φ_1 of (2.1) needs to be sought from particular solutions of the equation with $C_1 = \epsilon$:

$$\epsilon \Phi_1''' - \mathrm{i}(u-c)\,\Phi_1' + \mathrm{i}u'\Phi_1 = \mathrm{i}\epsilon. \tag{2.30}$$

The method of variation of constants provides a workable tool here also, since we have all the homogeneous solutions Φ_2 to Φ_4. According to the theorem cited, no differential equation needs to be solved, and a manipulation using the key property (2.28) leads to the following form of the solution:

$$\Phi_1 = \mathrm{i}\Phi_2 \int_{\hat{y}}^{y} w I_3\,\mathrm{d}y - \mathrm{i}\Phi_3 \int_0^y w I_2\,\mathrm{d}y + \mathrm{i}\Phi_4 \int_{\infty}^{y} w\,\mathrm{d}y, \tag{2.31}$$

with
$$I_l = \int^y \Phi_l w^{-2}\,\mathrm{d}y, \quad l = 2, 3, \tag{2.32}$$

where $y = \hat{y} \gg 1$ is a point beyond which the asymptotic expressions for Φ_2 to Φ_4 are valid. Taking derivatives successively, we have

$$\frac{\mathrm{d}^l \Phi_1}{\mathrm{d}y^l} = \mathrm{i}\frac{\mathrm{d}^l \Phi_2}{\mathrm{d}y^l}\int_{\hat{y}}^{y} w I_3\,\mathrm{d}y - \mathrm{i}\frac{\mathrm{d}^l \Phi_3}{\mathrm{d}y^l}\int_0^y w I_2\,\mathrm{d}y + \mathrm{i}\frac{\mathrm{d}^l \Phi_4}{\mathrm{d}y^l}\int_{\infty}^{y} w\,\mathrm{d}y + \mathrm{i}\delta_{3l}, \quad l = 0, 1, 2, 3, \tag{2.33}$$

a relationship easily checked using (2.29), (2.20) and (2.21).

The function Φ_1 in the form of (2.31) is shown to tend to a slowly varying function for $y \gg 1$ under the same conditions as invoked in deriving the asymptotic expression (2.19) for Φ_2; the resulting expression is

$$\Phi_1 \sim -(u-c)\int_{\hat{y}}^{y}(u-c)^{-2}\,\mathrm{d}y \quad (y > \hat{y} \gg 1). \tag{2.34}$$

For comparison, the classical equivalent of this function is noted:

$$(\Phi_1)_{\mathrm{CL,in}} = \tau + O(\alpha R)^{-\frac{1}{3}}, \quad (\Phi_1)_{\mathrm{CL,out}} = (u-c)\int^{y}(u-c)^{-2}\,\mathrm{d}y, \tag{2.35}$$

where τ has been defined in (2.13). Coincidence of our asymptotic expression with the classical *outer* solution is obvious here too.

In view of (2.33), we can easily show that the following relationship holds regarding the Wronskian formed by the four solutions Φ_1 to Φ_4:

$$\begin{aligned} w_{\mathrm{IV}} &= \begin{vmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \\ \Phi_1' & \Phi_2' & \Phi_3' & \Phi_4' \\ \Phi_1'' & \Phi_2'' & \Phi_3'' & \Phi_4'' \\ \Phi_1''' & \Phi_2''' & \Phi_3''' & \Phi_4''' \end{vmatrix} \\ &= -\mathrm{i}w_{\mathrm{III}} = -\mathrm{i}. \end{aligned} \tag{2.36}$$

As in (2.21), Φ_1 also is shown to obey a first-order differential equation. In fact, if we differentiate (2.31) and substitute (2.21) for Φ_2' and (2.29) for Φ_4' into the result, then we have the equation

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}y} - \lambda\Phi_1 = P, \tag{2.37}$$

with P defined by

$$P \equiv \mathrm{i}T\left[-\psi\hat{I}_3 - \int_{\hat{y}}^{y}\left(\Phi_2\Phi_3 - 2\int_{\infty}^{y}\Phi_3\frac{\mathrm{d}\Phi_2}{\mathrm{d}y}\mathrm{d}y\right)\frac{\mathrm{d}y}{\Phi_3 T^2}\right], \tag{2.38}$$

where

$$\psi = \int_{\infty}^{y} w\,\mathrm{d}y. \tag{2.39}$$

3. Higher-order approximations

Now that we have obtained the solution (1.6), namely the zero approximation of the power-series expansion in A^2, we can proceed to obtain the higher-order terms in the expansion (1.5). We will see *a posteriori* that the expansion (1.5) constitutes an alternating (complex) series; therefore convergence will be improved by introducing the parameter

$$\zeta = \frac{A^2}{\beta^2 + A^2}, \quad \text{where } \beta \text{ is a real constant}, \tag{3.1}$$

and by utilizing it as the expansion parameter to replace A^2. This is a generalization of the so-called Euler transformation which speeds up convergence of an alternating series. The classical Euler transformation corresponds to $\beta = A$.

We shall take advantage of this fact in advance and expand the Orr–Sommerfeld equation (1.1) in terms of ζ:

$$(\mathrm{L}_0 + \zeta\mathrm{L}_1 + \zeta^2\mathrm{L}_2)\, Y = 0, \tag{3.2}$$

where the operators L_0, L_1 and L_2 are shown to take the forms

$$\mathrm{L}_0 = \epsilon\,\mathrm{D}^4 - \mathrm{i}(u-c)\,\mathrm{D}^2 + \mathrm{i}u'', \tag{3.3}$$

$$\mathrm{L}_1 = -2\mathrm{L}_0 + \beta^2\,\mathrm{K}, \tag{3.4}$$

$$\mathrm{L}_2 = \mathrm{L}_0 - \beta^2\,\mathrm{K} + \epsilon\beta^4, \tag{3.5}$$

$$\mathrm{K} = \mathrm{i}(u-c) - 2\epsilon\,\mathrm{D}^2. \tag{3.6}$$

Let the eigenfunction be expanded in ζ accordingly:

$$Y = \Phi + \sum_{n=1}^{\infty} \zeta^n Y^{(n)}, \tag{3.7}$$

then substitution into (3.2) and rearrangement in the order of magnitude in ζ^n give us the following series of equations:

$$\mathrm{L}_0\,Y^{(n)} = -\mathrm{L}_1\,Y^{(n-1)} - \mathrm{L}_2\,Y^{(n-2)}, \quad n \geqslant 1, \tag{3.8}$$

with $Y^{(-1)}$ defined as zero. This equation, in turn, is written in an alternative form more convenient for actual calculations:

$$\mathrm{L}_0\,W^{(n)} = -\beta^2\,\mathrm{K}Z^{(n-1)} - \beta^4\epsilon\,Y^{(n-2)}, \quad n \geqslant 1, \tag{3.9}$$

where we have defined

$$Z^{(n)} = Y^{(n)} - Y^{(n-1)}, \tag{3.10}$$

$$W^{(n)} = Z^{(n)} - Z^{(n-1)} = Y^{(n)} - 2\,Y^{(n-1)} + Y^{(n-2)}. \tag{3.11}$$

Equation (3.9) is integrated once directly as in (1.7), and provides the following equation:

$$(\epsilon\,\mathrm{D}^3 - \mathrm{i}(u-c)\,\mathrm{D} + \mathrm{i}u')\,W^{(n)} = \mathrm{i}\epsilon(\beta^2\,\mathrm{L}Z^{(n-1)} + \beta^4\,\mathrm{M}\,Y^{(n-2)}), \tag{3.12}$$

where L and M are defined by

$$\mathrm{L}Z \equiv -\epsilon^{-1}\int_{\hat{y}}^{y} (u-c)\,Z\,\mathrm{d}y - 2\mathrm{i}Z', \tag{3.13}$$

and

$$\mathrm{M}\,Y \equiv \mathrm{i}\int_{\hat{y}}^{y} Y\,\mathrm{d}y. \tag{3.14}$$

Comparing this equation with (2.31), we see that it can be solved by the same method that was employed to obtain Φ_1 in §2. We then have, in parallel with (2.33),

$$\frac{\mathrm{d}^l W^{(n)}}{\mathrm{d}y^l} = \mathrm{i}\frac{\mathrm{d}^l\Phi_2}{\mathrm{d}y^l}\int_{\hat{y}}^{y} wI_3\,\Omega\,\mathrm{d}y - \mathrm{i}\frac{\mathrm{d}^l\Phi_3}{\mathrm{d}y^l}\int_{0}^{y} wI_2\,\Omega\,\mathrm{d}y$$

$$+\mathrm{i}\frac{\mathrm{d}^l\Phi_4}{\mathrm{d}y^l}\int_{\infty}^{y} w\,\Omega\,\mathrm{d}y + \mathrm{i}\delta_{3l}\,\Omega, \quad l = 0,1,2,3, \tag{3.15}$$

where Ω is defined by

$$\Omega \equiv \beta^2\,\mathrm{L}Z^{(n-1)} + \beta^4\,\mathrm{M}\,Y^{(n-2)}. \tag{3.16}$$

If this solution for W is substituted into (3.10) and (3.11) in which the variables $Z^{(n)}$ are eliminated, we obtain a formula for the eigenfunction Y in terms of Ws:

$$Y = \lim_{N\to\infty}\left[\frac{1-\zeta^N}{1-\zeta}\Phi + \sum_{m=1}^{N-1} k(\zeta, N-m)\,\zeta^m\,W^{(m+1)}\right], \tag{3.17}$$

where k is defined by

$$k(\zeta, N) \equiv \frac{1-(n+1)\,\zeta^n + n\zeta^{n+1}}{(1-\zeta)^2}. \tag{3.18}$$

4. The eigenvalue problem

In §2 we have obtained four independent solutions Φ_1 to Φ_4 at the level of zero approximation, or in the limit of vanishingly small wavenumber. If we also use one of these Φ_j as the Φ in (3.17), in the inhomogeneous terms $\Omega(\Phi)$ from which to calculate Ws successively, we will have four independent solutions Y_j $(j = 1, 2, 3, 4)$ with full wavenumber corrections incorporated. The fundamental properties by which the Φ_j are characterized (§2), namely that Φ_1 and Φ_2 are responsible for moderately varying functions, and Φ_3 and Φ_4 correspond to rapidly decaying and growing functions respectively, are preserved for the corresponding Y_j. In addition, the properties

$$\Phi_1(0) = \Phi_2(0) = \Phi_4(0) = 0, \quad \Phi_3(0) = 1 \tag{4.1}$$

that are directly checked in view of (2.31), (2.15), (2.24) and (2.3), are also shown to be preserved for Y_j:

$$Y_1(0) = Y_2(0) = Y_4(0) = 0, \quad Y_3(0) = 1. \tag{4.2}$$

In fact, all the correction terms $W^{(n)}$ in (3.17) vanish, as is readily seen from (3.15) (for $l = 0$) subject to (4.1).

With these solutions in hand, we can construct the general solution

$$Y = \sum_{j=1}^{4} C_j Y_j \tag{4.3}$$

for a semi-infinite shear flow bounded by a solid surface placed at $y = 0$ and unbounded otherwise. Then, of the four coefficients, C_4 should vanish to secure boundedness of Y as $Y \to \infty$. On the other hand, C_3 should also be zero in order for (4.3) to satisfy an impermeability condition at the wall,

$$Y(0) = C_1 Y_1(0) + C_2 Y_2(0) + C_3 Y_3(0) = C_3 = 0, \tag{4.4}$$

in view of (4.2). Thus it turns out that only Y_1 and Y_2 are needed to construct the eigenfunction:

$$Y = C_1 Y_1 + C_2 Y_2. \tag{4.5}$$

This is in contrast with the classical method, where three independent solutions are necessary for the same purpose. It does not mean, however, that the rapidly changing functions Φ_3 and Φ_4 are not involved at all in the present analysis. As is evident in view of the procedure leading to the solution Φ_2, we need to know Φ_3 (see (2.15)). The same is true for the solution Φ_1, because it depends on Φ_3 and Φ_4 through (2.31). This is a reflection of the fact that a rapidly changing nature is imbedded in Φ_1 and Φ_2 as well, although they are designed to behave as slowly varying only in the asymptotic limit.

The eigenvalue condition is formulated through imposing the no-slip condition at the wall,

$$Y'(0) = C_1 Y_1'(0) + C_2 Y_2'(0) = 0, \tag{4.6}$$

and forcing the solution to obey

$$Y' + AY = 0 \tag{4.7}$$

at a point $y = \hat{y}$, considered to be far enough away from the wall to obey the condition

$$A^2 \gg \left| \frac{u''}{u-c} \right|. \tag{4.8}$$

In fact, under this condition the Orr–Sommerfeld equation (1.1) has a solution decaying like e^{-Ay}, in other words, obeying (4.7). The requirement that the eigenfunction (4.5) should decay in this fashion far from the boundary reads

$$C_1(Y_1'+AY_1)+C_2(Y_2'+AY_2) = 0 \quad \text{at } y = \hat{y}. \tag{4.9}$$

This condition, together with condition (4.6), constitute an eigenvalue problem with the eigenvalue condition to be obtained from the following determinantal equation:

$$\begin{aligned} E(A,c,R) &\equiv E_r + \mathrm{i}E_i \\ &= \begin{vmatrix} Y_1'(0) & Y_2'(0) \\ Y_1'(\hat{y})+AY_1(\hat{y}) & Y_2'(\hat{y})+AY_2(\hat{y}) \end{vmatrix} \\ &= 0. \end{aligned} \tag{4.10}$$

This equation proves to be much simpler than its counterpart in the classical theory, and the determinant can be calculated directly through manipulating (3.17) for $\Phi = \Phi_1$ and Φ_2.

5. Calculations for the Blasius boundary layer: comparison with existing methods

As a step towards the future problem of determining stability characteristics of a wider class of flows, we will show, in this section, the stability characteristics of the Blasius profile calculated by the proposed method.

Figure 1 shows the elementary solutions Y_1 and Y_2 calculated up to the sixth approximation for Reynolds numbers $R = 2080$ and $R = 10^5$ respectively. Note that both functions satisfy the impermeability condition ($Y(0) = 0$), as they should. As is seen from this figure, the functions Y_1 and Y_2 are *not* slowly varying in the sense of classical asymptotic theory; they are rapidly varying near the boundary, and tend to be slowly varying functions only for $y \gg 1$. In fact, repeating the successive approximation (3.15) (with $l = 0$) which starts from (2.33) for Φ_1 and (2.15) for Φ_2, and carrying out the summation of the infinite series, we obtain

$$\left.\begin{aligned} Y_1 &\sim -(u-c)^{-1}\sinh A(y-\hat{y}), \\ Y_2 &\sim (u-c)\cosh A(y-\hat{y}) \end{aligned}\right\} \quad (y \gg \hat{y}).$$

Of course, these relationships hold only in the region where the rapidly changing factors (Φ_3 and Φ_4 in (2.31) and (2.15)) cease to operate.

Figure 2 shows the convergence check in terms of the first derivative of the eigenfunctions. Excellently rapid convergence is observed at $R = 10^5$, and the rate of convergence is reasonable at $R = 2080$. These results contrast sharply with the situation in the earlier computational method by which convergence gets more difficult as the Reynolds number is increased. Small ripples observed in crossing the critical layer $u = c$ for the fluctuation profile at $R = 10^5$ are seen to disappear at lower Reynolds numbers.

In figure 3 is shown the longitudinal fluctuation $|Y'|$ corresponding to the eigenvalues as listed in the figure. For comparison, results of calculations by Schlichting (1935), Radbil (1966) and Wazzan, Okamura & Smith (1968) at the same Reynolds number are also plotted. The measured eigenfunction from Schubauer & Skramstad (1948) is also shown for reference. Differences among the calculated eigenfunctions are hardly discernible, except for the result of Schlichting.

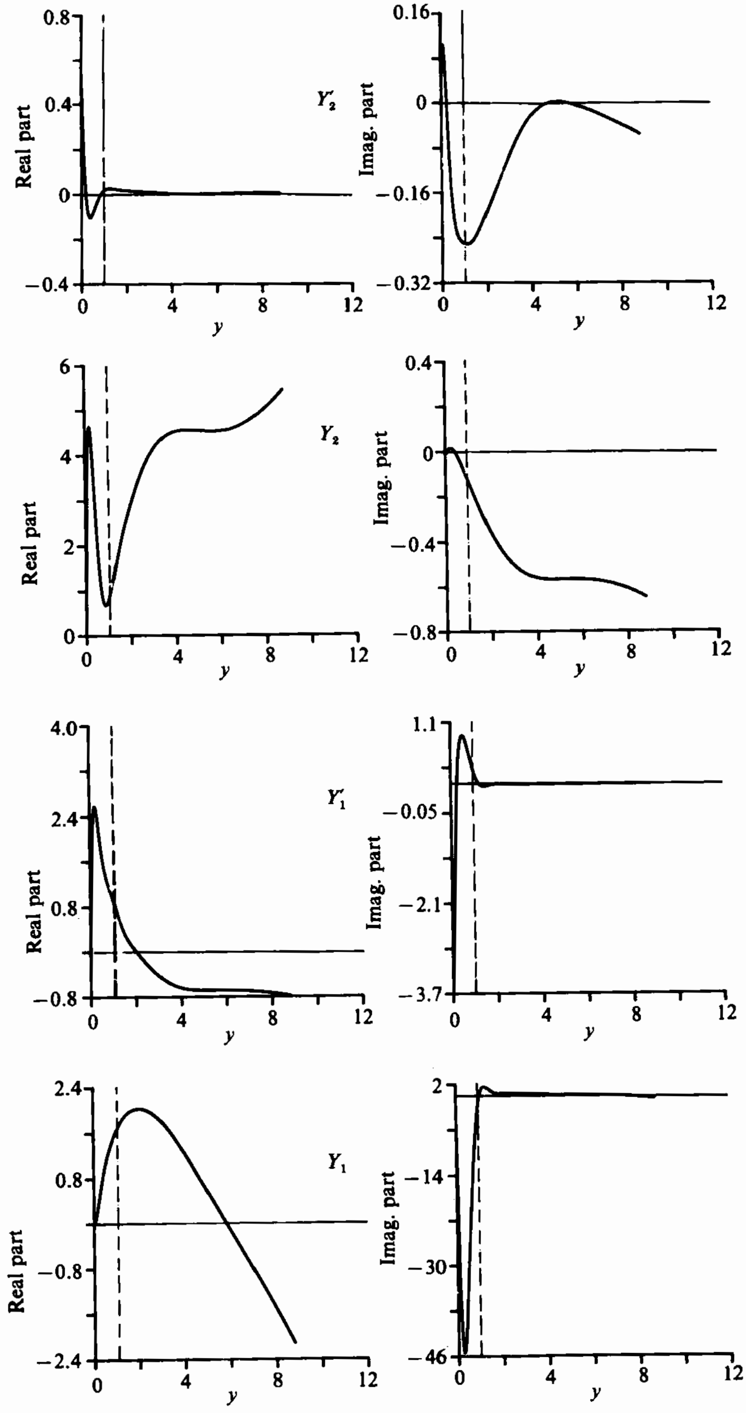

FIGURE 1(*a*). For description see opposite.

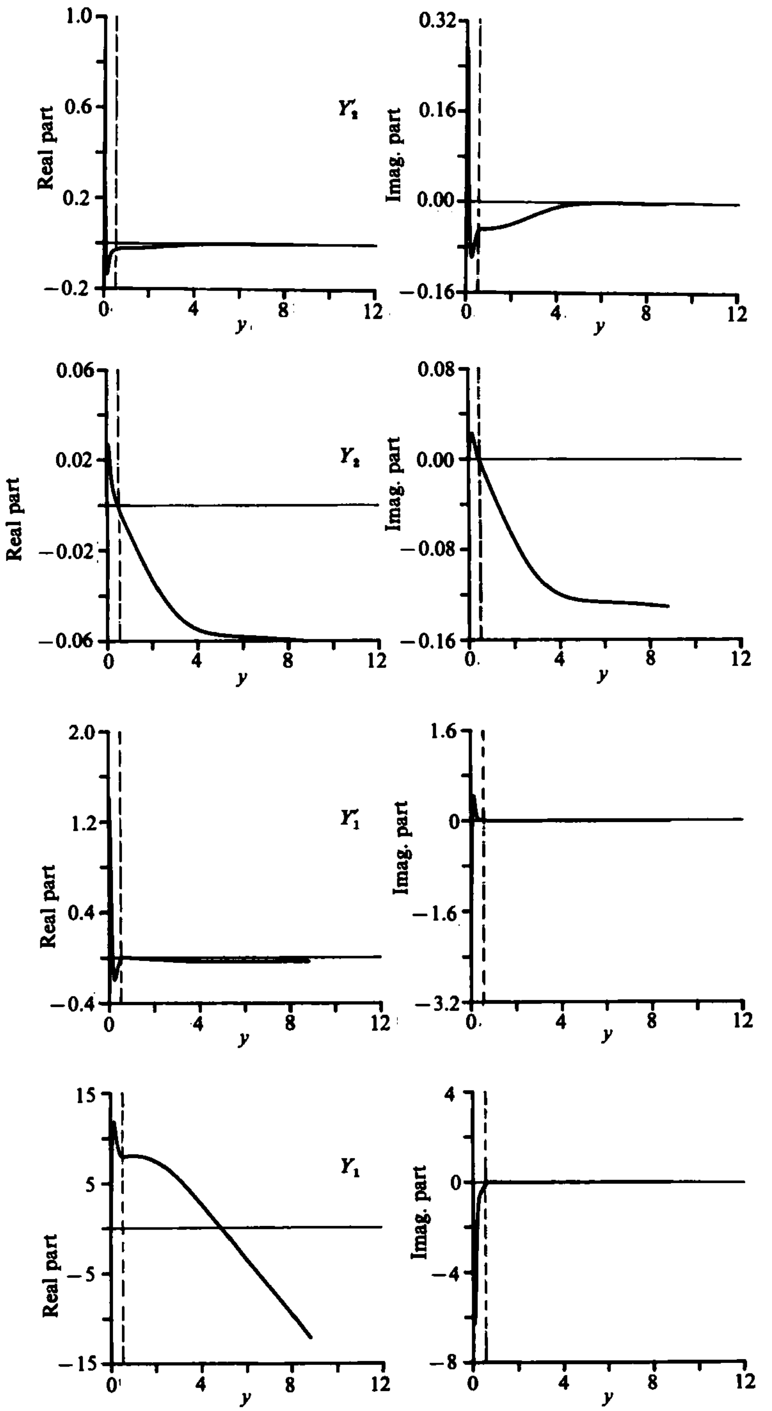

FIGURE 1. Elementary solutions Y_1 and Y_2 and their derivatives calculated up to the fifth approximation at Reynolds numbers $R = 2080$ (*a*) and 10^5 (*b*). Other parameters are so chosen as to satisfy the eigenvalue condition for neutral stability for the respective cases. The dotted lines denote the location of the critical layer ($u = c$).

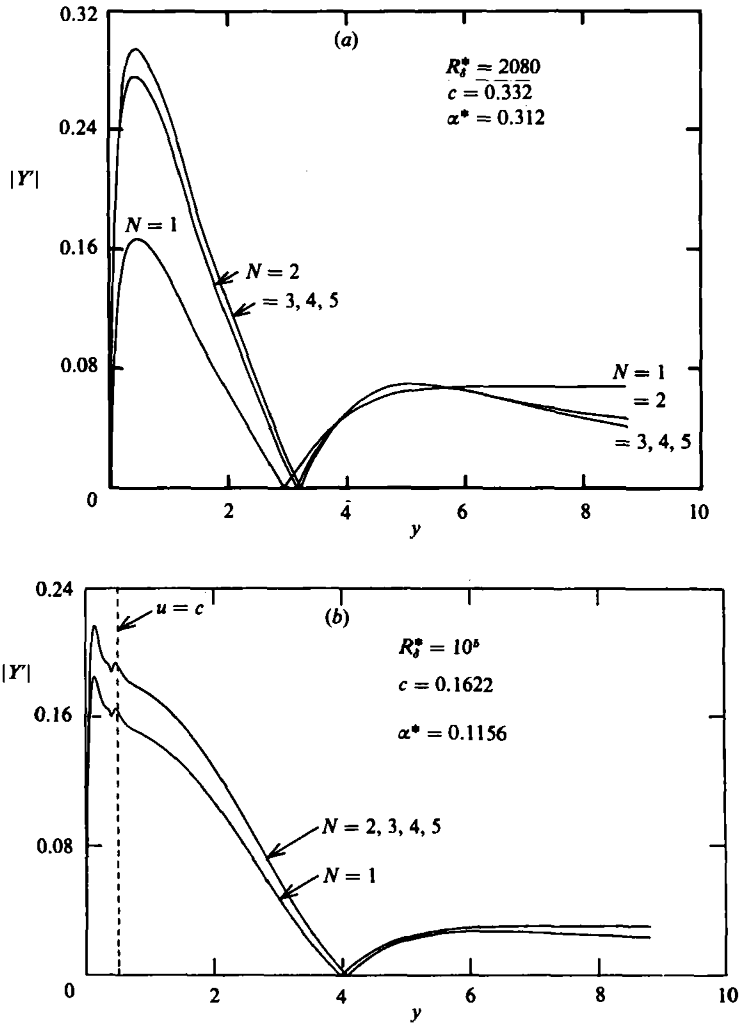

FIGURE 2. Convergence check at $R = 2080$ (a) and 10^6 (b) of the first derivative of the eigenfunction. The curves for $N = 5$ correspond to a linear combination of Y_1' and Y_2' of figure 1 to satisfy the no-slip condition.

Figure 4 shows the neutral-stability curve in the (α, R)-plane with the abscissa plotted against $R^{\frac{1}{3}}$ after Drazin and Reid (1981). The solid hairpin curve is the computational result due to Jordinson (1970). As the figure shows, its upper branch falls short of the classical asymptotic line of Lin (1945, 1946) because of the numerical stability limit. The eigenvalues calculated by the present method agree, within a difference not recognizable in the figure, with Jordinson's curve (the upper branch) in a Reynolds-number region from 5000 to the critical value $R = 519$. For higher Reynolds numbers the present method provides eigenvalues that merge with the asymptotic line. Figure 5 gives the same results plotted in the plane of phase velocity c and wavenumber α.

Beyond the value $R = 2.0 \times 10^5$ there is an indication of numerical instability observed in the solution procedure of (2.7). It seems to be caused by too rapid a change of λ in approaching the critical layer.

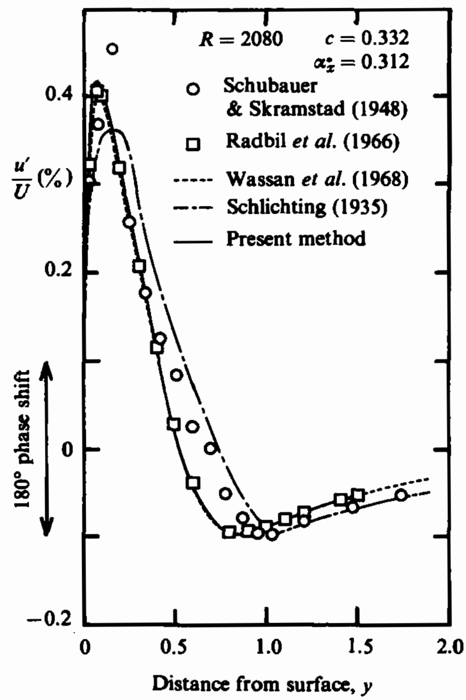

FIGURE 3. Comparison of the calculated eigenfunction with those by Schlichting (1935), Radbil (1966) and Wazzan *et al.* (1968) at the same Reynolds number of $R = 2080$. Experimental data by Schubauer–Skramstad are also plotted.

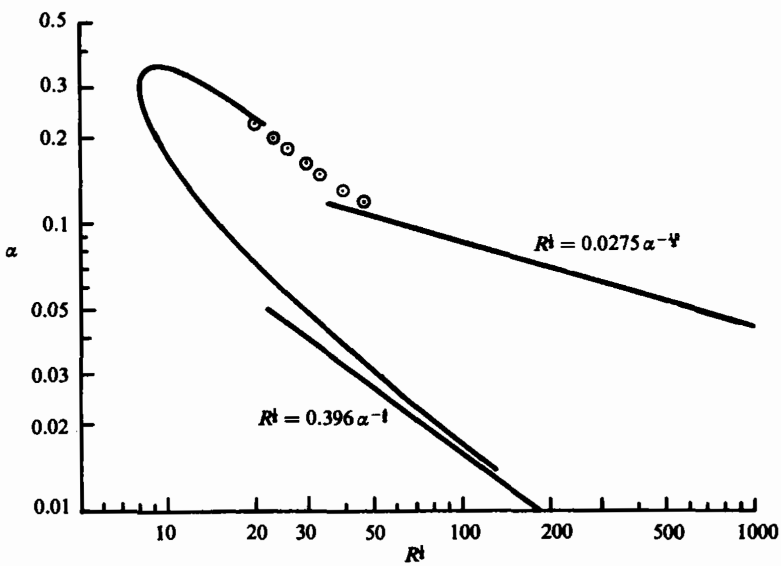

FIGURE 4. Neutral-stability curve in the (wavenumber, Reynolds number) plane of the Blasius profile. The calculated results are shown by dots. For Reynolds numbers smaller than 5000 the calculated points are not distinguishable from the solid hairpin curve due to Jordinson (1970). Two straight lines correspond to the asymptotic solution by Lin (1945, 1946).

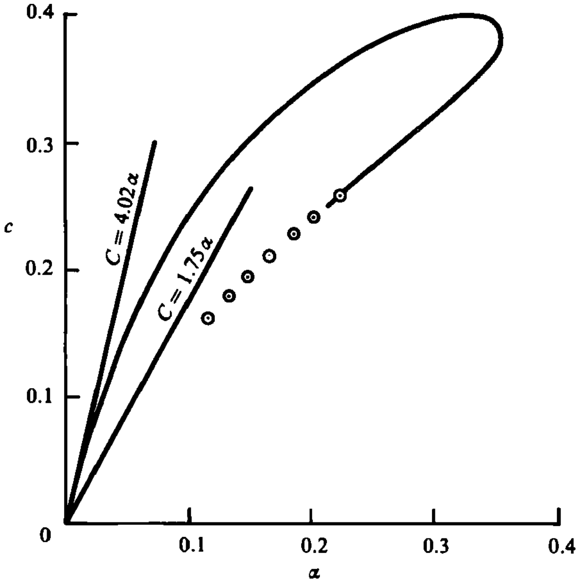

FIGURE 5. Neutral-stability curve in the (wave-velocity, wavenumber) plane, a representation based on the same data used in figure 4. The hairpin curve is due to Jordinson (1970) and the asymptotic lines are due to Lin (1945, 1946).

The high-Reynolds-number region as discussed here is of no practical interest for the Blasius boundary layer. However, for some Falkner–Skan boundary layers with favourable pressure gradients, the critical Reynolds number exceeds the value $R = 10^4$.

The authors have learned that the use of the compound matrix method in the numerical solution for obtaining the eigenfunction is successfully employed at an even higher Reynolds number of 10^6 (Davey 1982). We would like to note that the highest Reynolds number we have achieved is based on the use of a standard solver built in the available computer software, and can of course be improved.

6. Conclusions

An (almost) analytical method of solving the Orr–Sommerfeld equation that does not require the curvature data of the velocity profile is described and compared with existing methods. It is shown to have the advantage of simplicity in the formulation of the eigenvalue and the eigenfunction. It is also demonstrated that the method works for high Reynolds numbers where current computational methods fail. Applications to stability problems of shear flows with profiles given only through measurements or having no analytical expressions are suggested.

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